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Yu.R. Krakhmaleva
PhD in Technical Science
M.Kh.Dulaty Taraz University,
Taraz, Kazakhstan
yuna_kr@mail.ru.

A STUDY OF SERIES CONVERGENCE IN THE FIELD OF COMPUTATIONAL MATHEMATICS

Abstract. *Advances in modern computer technology have rendered it feasible to employ efficient methods, thereby reducing computational complexity. One effective method is the use of combined methods, which are implemented in computer algebra systems (CAS). This article delineates the evolution of code for the study of the convergence of series in the Maple CAS.*

Keywords: *series sum, code, series convergence, Maple, sufficient conditions for convergence.*

Introduction. The theory of series is a widely utilised research tool within the domain of theoretical natural sciences. The convergence property provides a robust foundation for the practical application of these concepts in applied problems. For instance, in approximate calculations, the approximation of functions and the calculation of their values is employed. In the theory of differential equations, a general solution is represented as a convergent power series over a specified interval.

Study Conditions and Methods. As is well known, an expression of the form:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \sum_{n=1}^{\infty} a_n \tag{1}$$

is a numerical sequence whose terms are the numbers a_n, where $n = \overline{1,\infty}$. In turn, a_n, referred to as the general term of the sequence, is written as a function of the number n. For application in problems of both practical and theoretical nature, the need for their convergence arises [1-3].

According to the theory of mathematical analysis, the convergence of the series follows from the existence of the sum of the series S in certain cases where the common term is sufficiently 'simple'. S:

$$\lim_{n\to\infty} S_n = S, \tag{2}$$

where S_n is the partial sum of the series. In such cases, finding the sum of the series depends directly on expressing S_n in a more compact form, suitable for finding its limit. In practice, however-specifically when dealing with a ‘complex’ formula such as a_n-one resorts to sufficient conditions for the convergence of numerical series. In the search for optimal and rational methods, we set ourselves the task of investigating the convergence of series in the Maple computer algebra system, which has a wealth of builtin commands, functions and packages for operations on series.

When solving problems involving numerical series, the general term of the series in Maple is treated as a term functionally dependent on n. To find the sum of the series, the command $sum(f(x,n), n=1..k)$. [4-5]. For example, let's calculate the S_n and S terms of the sequence:

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)\cdot(n+2)}.$$

$$SR := sum\left(\frac{1}{n\cdot(n+1)\cdot(n+2)}, n=1..k\right);\ S := sum\left(\frac{1}{n\cdot(n+1)\cdot(n+2)}, n=infinity\right);$$

$$SR := -\frac{1}{2\,(k+1)}+\frac{1}{2\,(k+2)}+\frac{1}{4}$$

$$S := 0$$

As can be seen, SR is the partial sum of the sequence; the numbering started at $n=\overline{1,k}$, and the result displayed is its representation. When finding the sum of the series S, the numbering changed to $n=\overline{1,\infty}$ and the result we obtained was the sum of the series $S=0$. Thus, there is no need to use the formula (2).

Let us examine the convergence of the series $\sum_{n=1}^{\infty}\frac{(n+1)!}{(n+2)5^n}$ by applying D'Alembert's sufficient condition for convergence:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{(n+2)!}{(n+3)5^{n+1}}\cdot\frac{(n+2)5^n}{(n+1)!}=\lim_{n\to\infty}\frac{(n+2)(n+1)!}{(n+3)5^{n+1}}\cdot\frac{(n+2)5^n}{(n+1)!}=$$

$$=\lim_{n\to\infty}\frac{(n+2)}{(n+3)}\cdot\frac{5^n}{5^{n+1}}=\lim_{n\to\infty}\frac{(n+2)}{(n+3)}\cdot\frac{1}{5}=\infty$$

This means that the series diverges. Let us now examine the series for convergence in Maple: we will apply D'Alembert's test and, to verify the result, calculate the sum S [4-5]:

$$Limit\left(\frac{\frac{(n+2)!}{(n+3)\cdot 5^{n+1}}}{\frac{(n+1)!}{(n+2)\cdot 5^n}}, n=infinity\right)=limit\left(\frac{\frac{(n+2)!}{(n+3)\cdot 5^{n+1}}}{\frac{(n+1)!}{(n+2)\cdot 5^n}}, n=infinity\right);$$

$$SS := sum\left(\frac{(n+1)!}{(n+2)\cdot 5^n}, n=1..infinity\right);$$

$$\lim_{n\to\infty}\frac{(n+2)!\,(n+2)\,5^n}{(n+3)\,5^{n+1}\,(n+1)!}=\infty$$

$$SS := \infty$$

As can be seen, the value of the SS term of the series confirms that the convergence criterion is not satisfied.

Let us examine the convergence of alternating series, the full analysis of which involves testing for absolute and conditional convergence.[6]-[7] Applying the theory of mathematical analysis, we will examine an alternating series, for example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{n+1}$$

According to Leibniz's criterion, we have:

$$1.\ \frac{\sqrt{1}}{2} < \frac{\sqrt{2}}{3} < \frac{\sqrt{3}}{4} < \ldots < \frac{\sqrt{n}}{n+1},$$

$$2.\ \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0,$$

which implies that the series under consideration converges. Let us continue the proof and examine the series for absolute convergence:

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

then $\frac{\sqrt{n}}{n+1} \approx \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Let's compare it with a harmonic series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which, as is well known, diverges, since. $p = \frac{1}{2} < 1$.

Using the limit test, we obtain:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n} \cdot \sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ also diverges, which implies that the original series is conditionally convergent $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{n+1}$.

Using the results obtained from the study of the convergence of numerical series, we will examine the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{n+1}$ for absolute convergence in Maple; specifically, without using Leibniz's test, we will limit ourselves to finding the sum of the series:

$$S1 := sum\left(\frac{(-1)^n \cdot \mathrm{sqrt}(n)}{n}, n = 1 .. infinity\right); \quad S2 := sum\left(\frac{\mathrm{sqrt}(n)}{n}, n = 1 .. infinity\right);$$

$$S1 := \mathrm{polylog}\left(\frac{1}{2}, -1\right)$$

$$S2 := \infty$$

As can be seen, the series has a sum of . However, the sum of the series, taken modulo $\sum_{n=1}^{\infty}\left|\frac{(-1)^n\cdot\sqrt{n}}{n+1}\right|$, is ∞; that is, the series does not converge.

From the examples considered, the following conclusion can be drawn: In Maple, to test the convergence of numerical and alternating series, it is sufficient to use the command $sum(f(x,n), n=1..k)$, which computes the sum S of the series under investigation. If, as a result of executing this command, the value S of the series is a real number, then the series converges. If the value is $S=\infty$, then the series diverges. The code for testing the convergence of numerical series with non-negative terms is as follows:

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$$SR := sum\left(\frac{(n+1)!}{(n+2)\cdot 5^n}, n = 1 .. infinity\right);$$

```
if SR = infinity then print ('Ряд_расходится');
else print (Ряд_ссходится );
fi;
```

$$SR := \infty$$

print Ряд_расходится

The formula for the convergence test of alternating series with non-negative terms is as follows:

$$S1 := sum\left(\frac{(-1)^n\cdot sqrt(n)}{n}, n = 1 .. infinity\right);$$

$$S2 := sum\left(\operatorname{abs}\left(\frac{(-1)^n\cdot sqrt(n)}{n}\right), n = 1 .. infinity\right);$$

```
if S1 = infinity then print ('Ряд_S1_расходится');
else print (Ряд_S1_сходится );
fi;
 if S2 = infinity then print ('Ряд_S1_сходится условно');
else print (Ряд_S1_сходится _абсолютно);
fi;
```

$$S1 := \operatorname{polylog}\left(\frac{1}{2}, -1\right)$$

$$S2 := \infty$$

print Ряд_S1_сходится

print Ряд_S1_сходится условно

Functional series are of great practical use; when investigating them, the question of convergence is supplemented by determining the region of convergence. Let us consider an example of investigating a functional series.[2]

Example 1. Investigate the convergence of the following functional series:

$$\sum_{n=1}^{\infty}\frac{(3n^2+5)4^n}{(6x-5)^n}.$$

Given the general term of the sequence, D'Alembert's rule applies:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{(3(n+1)^2+5)4^{n+1}}{(6x-5)^{n+1}}}{\dfrac{(3x^2+5)4^n}{(6x-5)^n}}\right|=\lim_{n\to\infty}\left|\frac{\left(3(n+1)^2+5\right)4^n}{(6x-5)\left(3x^2+5\right)}\right|=$$

$$=\frac{4}{|6x-5|}\lim_{n\to\infty}\frac{3(n+1)^2+5}{3n^2+5}=\frac{4}{|6x-5|}$$

If condition $\frac{4}{|6x-5|}<1$ is satisfied, the series converges; by solving the inequality, we obtain its region of convergence: $1/6<x<3/2$. Upon examining the series at the boundary points of the region of convergence, we have: $x=\frac{1}{6}$; the series is numerical and alternating, and takes the form $\sum_{n=1}^{\infty}(-1)^n\left(4n^2+5\right)$. By Leibniz's test, the series diverges; moreover, the series converges conditionally. For $x=\frac{3}{2}$, the series is numerical with non-negative terms and is written in the form $\sum_{n=1}^{\infty}\left(4n^2+5\right)$. For the series, the necessary convergence criterion is not satisfied: $\lim_{n\to\infty}\left(4n^2+5\right)\neq 0$, which means we cannot apply the sufficient convergence criteria. Thus, the region of convergence of the series is: $\left(-\infty;\frac{1}{6}\right)\cup\left(\frac{3}{2};+\infty\right)$.

Let us compute the region of convergence of the series $\sum_{n=1}^{\infty}\frac{(3n^2+5)4^n}{(6x-5)^n}$ in Maple. We will apply D'Alembert's criterion[4, 5]:

$$Limit\left(\frac{\dfrac{\left(3\cdot(n+1)^2+5\right)\cdot 4^{(n+1)}}{\mathrm{abs}(6\cdot x-5)^{(n+1)}}}{\dfrac{\left(3\cdot n^2+5\right)\cdot 4^n}{\mathrm{abs}(6\cdot x-5)^n}}, n=infinity\right)=limit\left(\frac{\dfrac{\left(3\cdot(n+1)^2+5\right)\cdot 4^{(n+1)}}{\mathrm{abs}(6\cdot x-5)^{(n+1)}}}{\dfrac{\left(3\cdot n^2+5\right)\cdot 4^n}{\mathrm{abs}(6\cdot x-5)^n}}, n=infinity\right);$$

$$\lim_{n\to\infty}\frac{\left(3(n+1)^2+5\right)4^{n+1}|6x-5|^n}{|6x-5|^{n+1}\left(3n^2+5\right)4^n}=\frac{4}{|6x-5|}$$

Let us solve the inequality with the convergence condition based on the following criterion: $req := solve\left(\frac{4}{\text{abs}(6\cdot x-5)} < 1, x\right);$

$$req := RealRange\left(Open\left(\frac{3}{2}\right), \infty\right), RealRange\left(-\infty, Open\left(\frac{1}{6}\right)\right)$$

The solution found for the interval is the region of convergence of the series: $\left(-\infty;\frac{1}{6}\right)\cup\left(\frac{3}{2};+\infty\right)$. Let us examine the behaviour of the series at the endpoints of the convergence region. We shall substitute the endpoints of the intervals for x in the original series. For $x=\frac{1}{6}$, we have: $s1 := sum\left(\frac{(3\cdot n^2+5)\cdot 4^n}{\left(6\cdot\frac{1}{6}-5\right)^n}, n = 1..infinity\right);$

$$s1 := \sum_{n=1}^{\infty}\frac{(3\,n^2+5)\,4^n}{(-4)^n}$$

Maple evaluates the series, which means that the programme cannot perform any transformations on the number (-4). However, from the result, we can see that the series is alternating. A negative value appears in the denominator: $\left(6\cdot\frac{1}{6}-5\right)$. Let us rewrite the series in a different form, taking this fact into account:

$$s11 := sum\left(\frac{(3\cdot n^2+5)\cdot 4^n}{\text{abs}\left(\left(6\cdot\frac{1}{6}-5\right)\right)^n}, n = 1..infinity\right);$$

$$s11 := \infty$$

When this entry was made, the programme returned the result: $S=\infty$. This means that the series diverges at the point $x=\frac{1}{6}$.

Furthermore, at $x=\frac{3}{2}$ we have:

$$s2 := sum\left(\frac{(3\cdot n^2+5)\cdot 4^n}{\left(6\cdot\frac{3}{2}-5\right)^n}, n = 1..infinity\right);$$

$$s2 := \infty$$

The term in the denominator is positive and the series converges at $x=\frac{3}{2}$. $S=\infty$. It follows that the series diverges at the point. $x=\frac{3}{2}$ Therefore, the region of convergence is: $\left(-\infty;\frac{1}{6}\right)\cup\left(\frac{3}{2};+\infty\right)$

Let us consider the special case of a power series where the region of convergence is an interval of convergence. The interval of convergence of the power series

$\sum_{n=0}^{\infty} a_n x^n$ is the set of values x at which the series converges absolutely.

Example 2. Investigate the convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 5^n}$.

Let us apply D'Alembert's test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \frac{|x+1|^{n+1} \cdot 5^n \cdot n}{(n+1) \cdot 5^{n+1} \cdot |x+1|^n} =$$

$$= |x+1| \cdot \lim_{n\to\infty} \frac{n}{5(n+1)} = \frac{|x+1|}{5}$$

From $\frac{|x+1|}{5} < 1$, we obtain $|x+1| < 5$. Consequently, $-6 < x < 4$ is a convergence interval for $R = 5$. We can also examine the endpoints of the interval, namely:

a) $x = 4$, $u_n(4) = \frac{1}{n}$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges;

b) $x = -6$, $u_n(-6) = \frac{(-1)^n}{n}$,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges conditionally.

Similarly, $[-6; 4)$, $R = 5$ constitutes a convergence region.

Let us examine the convergence of the series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 5^n}$ in Maple. Let us execute the code commands:

$$Limit\left(\frac{\frac{(\operatorname{abs}(x+1))^{(n+1)}}{(n+1) \cdot 5^{n+1}}}{\frac{(\operatorname{abs}(x+1))^n}{n \cdot 5^n}}, n = infinity \right) = limit\left(\frac{\frac{(\operatorname{abs}(x+1))^{(n+1)}}{(n+1) \cdot 5^{n+1}}}{\frac{(\operatorname{abs}(x+1))^n}{n \cdot 5^n}}, n = infinity \right);$$

$$\lim_{n\to\infty} \frac{|x+1|^{n+1}\, n\, 5^n}{(n+1)\, 5^{n+1}\, |x+1|^n} = \frac{1}{5} |x+1|$$

$$req := solve\left(\frac{\operatorname{abs}(x+1)}{5} < 1, x \right);$$

$$req := RealRange(Open(-6), Open(4))$$

The region of convergence of the series: $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 5^n}$: $(-6; 4)$. Let us examine the behaviour of the series at the boundaries of the region:

$$s1 := sum\left(\frac{(-6+1)^n}{n \cdot 5^n}, n = 1 .. infinity\right);\ s2 := sum\left(\frac{(4+1)^n}{n \cdot 5^n}, n = 1 .. infinity\right);$$

$$s1 := -\ln(2)$$

$$s2 := \infty$$

The fact that the sum is $s_1 = -\ln 2$ means that the series converges at the point $x = -6$. At the point $x = 4$, however, the series diverges, since $s_2 = \infty$. This gives the region of convergence of the series: $s_2 = \infty$.

When investigating functional and also power series to determine the region of convergence, well-known sufficient criteria for convergence are applied.[6]-[7] When examining the behaviour of the series at the boundary points of the region of convergence, the command $sum(f(x,n), n = 1..k)$ is used. It is difficult to predict the behaviour of a series at various points where the variable changes; therefore, Maple commands are used sequentially when solving problems involving the convergence of functional series, building on the result of the previous command[8].

Research results. An investigation of the convergence of series in the computer algebra system Maple has shown that, for the convergence of numerical and alternatingseries with non-negative terms, it is sufficient to find the sum of the series without resorting to sufficient convergence criteria. This development enabled the creation of automated convergence testing code for these series.

The study of functional and power series is conducted in several stages. In the initial phase, adequate convergence criteria are implemented to ascertain the convergence region of the series. In the subsequent phase, a refinement of the convergence region of the series is undertaken. In the final stage, an analysis is performed at the boundary points of the resulting convergence region, where automated convergence analysis code is applied to numerical and alternating series with non-negative terms.

Discussion of Research Results. The development of convergence analysis code for numerical and alternating series with non-negative terms has been enabled by numerous practical examples, including the implementation of the solution in the Maple system. The code facilitates the execution of the study without the necessity of adhering to sufficient convergence criteria, which are applied in accordance with the form of the general term of the series. Consequently, the number of computational operations is reduced, thereby minimising the computation process. The simplicity of the convergence analysis code is indisputable.

In the course of examining the convergence of functional and power series, the development of the code encounters challenges that are predominantly associated with the selection of an adequate convergence criterion, which is based on the structure of the general term of the series. In this instance, an algorithm is presented for investigating the convergence of functional series and determining the convergence region.

Conclusion. The implementation of combined methods in computer mathematics systems engenders a broad spectrum of potential applications in various fields of enquiry. The transparency of the operations performed, the speed of computation, and the ability to make adjustments all serve to increase the effectiveness of solution methods.

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Ю.Р Крахмалева

Таразский университет имени М.Х. Дулати

Тараз, Казахстан

Исследование сходимости рядов в системе компьютерной математики

Аннотация. Развитие современных компьютерных технологий дает возможность применения рациональных методов за счет снижения трудоемкости вычислений. Одним из эффективных методов рассматривается применение комбинированных методов, которые реализуются в системах компьютерной математики (СКМ). В статье показана разработка кода исследования сходимости рядов в СКМ Maple.

Ключевые слова: сумма ряда, код, сходимость ряда, Maple, достаточные признаки сходимости

Ю.Р. Крахмалева

М.Х. Дулати атындағы Тараз университеті

Тараз, Казахстан

Компьютерлік математика жүйесіндегі қатарлардың жинақтылығын зерттеу

Аннотация. Қазіргі заманғы компьютерлік технологиялардың дамуы есептеулердің еңбек сыйымдылығын төмендету арқылы тиімді әдістерді қолдануға мүмкіндік береді. Тиімді әдістердің бірі ретінде компьютерлік математика жүйелерінде (КМЖ) жүзеге асырылатын аралас әдістерді қолдану қарастырылады. Мақалада Maple жүйесінде қатарлардың жинақтылығын зерттеуге арналған кодты әзірлеу көрсетілген.

Кілт сөздер: қатардың қосындысы, код, қатар жинақтылығы, Maple, жинақтылықтың жеткілікті белгілері

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